Exercise 54

Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

Solution

Calculate the derivative of this function using the product rule.

$$\begin{aligned}
\frac{d^9}{dx^9}(x^8 \ln x) &= \frac{d^8}{dx^8}\left\{\left[\frac{d}{dx}(x^8)\right]\ln x + x^8\left[\frac{d}{dx}(\ln x)\right]\right\} \\
&= \frac{d^8}{dx^8}\left[(8x^7)\ln x + x^8\left(\frac{1}{x}\right)\right] \\
&= \frac{d^8}{dx^8}(8x^7 \ln x + x^7) \\
&= \frac{d^7}{dx^7}(56x^6 \ln x + 8x^6 + 7x^6) \\
&= \frac{d^7}{dx^7}(56x^6 \ln x + 15x^6) \\
&= \frac{d^6}{dx^6}(336x^5 \ln x + 56x^5 + 90x^5) \\
&= \frac{d^6}{dx^6}(336x^5 \ln x + 146x^5) \\
&= \frac{d^5}{dx^5}(1680x^4 \ln x + 336x^4 + 730x^4) \\
&= \frac{d^5}{dx^5}(1680x^4 \ln x + 1066x^4) \\
&= \frac{d^4}{dx^4}(6720x^3 \ln x + 1680x^3 + 4264x^3) \\
&= \frac{d^4}{dx^4}(6720x^3 \ln x + 5944x^3) \\
&= \frac{d^3}{dx^3}(20160x^2 \ln x + 6720x^2 + 17832x^2) \\
&= \frac{d^3}{dx^3}(20160x^2 \ln x + 24552x^2) \\
&= \frac{d^2}{dx^2}(40320x \ln x + 20160x + 49104x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^9}{dx^9}(x^8 \ln x) &= \frac{d^2}{dx^2}(40320x \ln x + 69264x) \\
&= \frac{d}{dx}(40320 \ln x + 40320 + 69264) \\
&= \frac{d}{dx}(40320 \ln x + 109584) \\
&= 40320 \left[\frac{d}{dx}(\ln x)\right] + \frac{d}{dx}(109584) \\
&= 40320 \left(\frac{1}{x}\right) + 0 \\
&= \frac{40320}{x}.
\end{aligned}$$